Lecture 2: Main Ingridients of QCD SRs
Dispersion relations, condensates, OPE

A. P. Bakulev

Bogolyubov Lab. Theor. Phys., JINR (Dubna, Russia)

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Содержание 2-ой лекции

- Дисперсионные соотношения в векторном канале: почему в корреляторе T-произведение?
- Дуальность кварки–адроны: глобальная и локальная.
- Преобразование Бореля.
- Конденсаты кварковых и глюонных полей.
- Операторное разложение: скалярная модель и КХД

Dispersion Relations
in
vector channel

$\Pi_{\mu\nu}$- и $\Pi^{+}_{\mu\nu}$-кореляторы

Наш коррелятор определен через T-произведение:

$$\Pi_{\mu\nu}(q) = i \int d^4x \, e^{iqx} \langle 0 \big| T \left[J_\mu(x) J_\nu(0) \right] \big| 0 \rangle \, .$$

Что мы знаем о лоренцевой структуре $\Pi_{\mu\nu}(q^2)$?

$\Pi_{\mu\nu}$- *и* $\Pi^{+}_{\mu\nu}$-*кореляторы*

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Что мы знаем о лоренцевой структуре $\Pi_{\mu\nu}(q^2)$?
Рассмотрим сначала структуру

$$\Pi^{+}_{\mu\nu}(q) = \int d^4x\, e^{iqx} \langle 0 \big| J_\mu(x) J_\nu(0) \big| 0 \rangle \,.$$

и ее свертку по лоренцевым индексам

$$\Pi^{+}(q) = \frac{i^2}{3} \int d^4x\, e^{iqx} \langle 0 \big| J_\mu(x) J^\mu(0) \big| 0 \rangle \,.$$

Спектральная плотность $\Pi^{+}_{\mu\nu}$-корелятора

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Вставим между токов $\hat{1} = \sum_{X(p)}\big|X(p)\rangle\langle X(p)\big|$:

$$\Pi^{+}(q) = \frac{i^2}{3}\sum_{X(p)}\theta(p_0)\int d^4x\, e^{iqx}\langle 0\big|J_\mu(x)\big|X(p)\rangle\langle X(p)\big|J^\mu(0)\big|0\rangle$$

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и воспользуемся свойством

$$\langle 0\big|J_{\mu}(x)\big|X(p)\rangle = e^{-ipx}\langle 0\big|J_{\mu}(0)\big|X(p)\rangle\,.$$

Спектральная плотность $\Pi^{+}_{\mu\nu}$*-корелятора*

В результате получим $\boxed{\Pi^{+}(q) = 2\pi\, q^2\, \theta(q_0)\, \rho(q^2)}$, где

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2} \sum_{X(p)} \delta^{(4)}(q-p) \left|\langle 0 \big| J_\mu(0) \big| X(p)\rangle\right|^2 .$$

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Давайте убедимся, что $\boxed{\rho(q^2) \geq 0}$.

Спектральная плотность $\Pi^{+}_{\mu\nu}$-корелятора

As a result we have $\boxed{\Pi^{+}(q) = 2\pi\, q^2\, \theta(q_0)\, \rho(q^2)}$, where

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Lorentz-invariance dictates

$$\langle 0 \big| J_\mu(x) \big| X(p) \rangle = \left[A(p)\, p_\mu + B(p)\, \varepsilon_\mu\right] e^{-ipx}$$

with $p \cdot \varepsilon = 0$.

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with $p \cdot \varepsilon = 0$, and therefore $\varepsilon \cdot \varepsilon = -1$. From current conservation it follows $A(p) = 0$, i. e.

$$\langle 0 | J_\mu(x) | X(p)\rangle \langle X(p) | J_\mu(x) | 0\rangle = \left|B(p)\right|^2 \varepsilon^2 = -\left|B(p)\right|^2 \leq 0 .$$

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that gives us

$$\rho(q^2) \geq 0$$

Связь с сечением $e^+e^- \to$ *адроны*

Итак, $\boxed{\Pi^+(q) = 2\pi\, q^2\, \theta(q_0)\, \rho(q^2)}$, где

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Важно, что эта функция естественно возникает в описании процесса $e^+e^- \to$ адроны
в 1-фотонном приближении КЭД:

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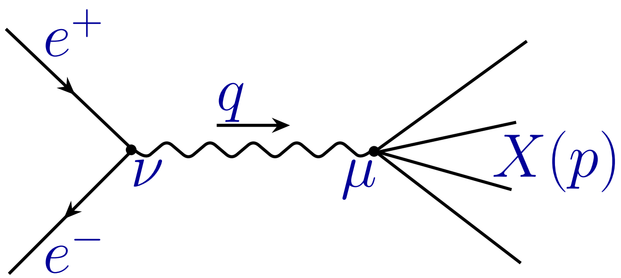

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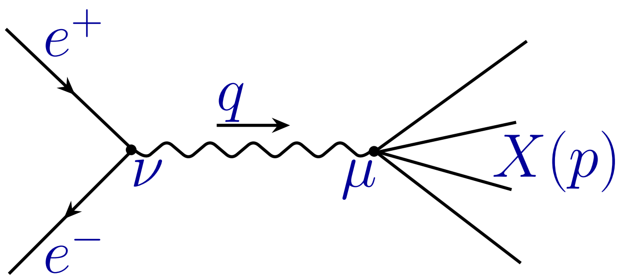

$$\bar{u}(k)\gamma_\mu u(k')\frac{ie^2}{q^2}\langle X(p)|J_\mu(q)|0\rangle$$

Связь с сечением $e^+e^- \to$ адроны

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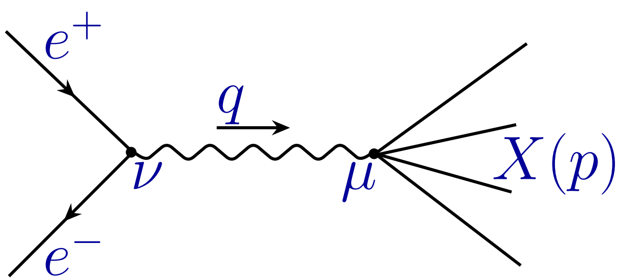
Important that this function naturally appears in description of the process $e^+e^- \to$ hadrons in 1-photon approximation of QED:



$$\sigma_{\text{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\,\rho(s)$$

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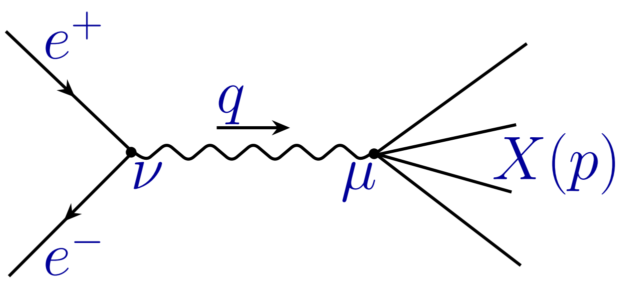
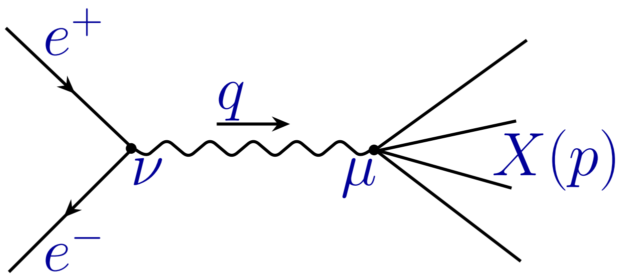


$$\boxed{\sigma_{\text{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\,\rho(s) = \frac{4\,\pi\,\alpha^2}{3\,s}\,R(s)}$$

Связь с сечением $e^+e^- \to$ адроны

In 1-photon approximation of QED:



$$\sigma_{\text{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\,\rho(s) = \frac{4\,\pi\,\alpha^2}{3\,s}\,R(s)$$

Here we explicitly extract as a factor cross-section $\sigma_{\mu^+\mu^-}(s) = 4\,\pi\,\alpha^2/(3\,s)$ of the process $e^+e^- \to \mu^+\mu^-$.

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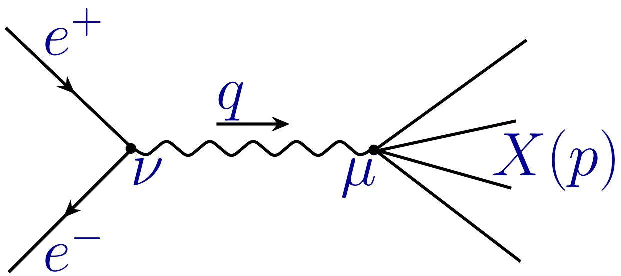

$$\sigma_{\text{had}}(s) \;=\; \frac{16\,\pi^3\,\alpha^2}{s}\,\rho(s) = \frac{4\,\pi\,\alpha^2}{3\,s}\,R(s)$$

Here we explicitly extract as a factor cross-section $\sigma_{\mu^+\mu^-}(s) = 4\,\pi\,\alpha^2/(3\,s)$ of the process $e^+e^- \to \mu^+\mu^-$.
Therefore,

$$R(s) \;=\; \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Спектральная плотность $\Pi^{+}_{\mu\nu}$-корелятора

Из релятивистской инвариантности и сохранения векторного тока мы получаем

$$\Pi^{+}_{\mu\nu}(q) = \left[\frac{q_\mu\, q_\nu}{q^2} - g_{\mu\nu}\right] \Pi^{+}(q)$$

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Spectral density of $\Pi_{\mu\nu}$-correlator

Relativistic covariance + vector current conservation gives us

$$\Pi^{+}_{\mu\nu}(q) = \left[\frac{q_\mu\, q_\nu}{q^2} - g_{\mu\nu}\right] \Pi^{+}(q) = \left[q_\mu\, q_\nu - g_{\mu\nu}\, q^2\right] 2\pi\, \theta(q_0)\, \rho(q^2)\,.$$

Our $\Pi_{\mu\nu}$-correlator is defined using T-product:

$$\Pi_{\mu\nu}(q) = i \int d^4x\, e^{iqx} \langle 0 \big| T\left[J_\mu(x) J_\nu(0)\right] \big| 0 \rangle\,,$$

it also has an evident representation

$$\Pi_{\mu\nu}(q) = \left[q_\mu\, q_\nu - g_{\mu\nu}\, q^2\right]\, \Pi(q)\,.$$

Spectral density of $\Pi_{\mu\nu}$-correlator

Inserting $\hat{1}$ in between currents we obtain

$$\Pi(q) = \frac{-i}{3q^2} \sum_{X(p)} \int_0^\infty dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J_\mu(x) \big| X(p) \rangle \langle X(p) \big| J^\mu(0) \big| 0 \rangle$$

$$+ \frac{-i}{3q^2} \sum_{X(p)} \int_{-\infty}^0 dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J_\mu(0) \big| X(p) \rangle \langle X(p) \big| J^\mu(x) \big| 0 \rangle$$

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$$= \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p} - \vec{q})\, \theta(p_0) \left| \langle 0 \big| J_\mu(0) \big| X(p) \rangle \right|^2$$

$$\times \int_0^{\infty} dt \left[e^{i(q_0 - p_0)t} + e^{-i(q_0 + p_0)t} \right]$$

Spectral density of $\Pi_{\mu\nu}$*-correlator*

Or

$$\Pi(q^2) = \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q}) \left|\langle 0\right| J_\mu(0) \left|X(p)\rangle\right|^2 \int_0^\infty dt\, e^{i(|q_0|-p_0)t}\,.$$

We have the following identities

$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\,\delta(\alpha) \pm i\,\mathcal{P}\frac{1}{\alpha}\,.$$

Spectral density of $\Pi_{\mu\nu}$*-correlator*

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We have the following identities

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After all substitutions we obtain

$$\mathbf{Im}\,\Pi(q^2) = -\pi\,\frac{(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q})\,\delta(p_0-|q_0|) \left|\langle 0\big|J_\mu(0)\big|X(p)\rangle\right|^2$$

Spectral density of $\Pi_{\mu\nu}$*-correlator*

В результате всех подстановок получим

$$\mathbf{Im}\,\Pi(q^2) = -\pi\,\frac{(2\pi)^3}{3q^2}\sum_{X(p)}\delta(\vec{p}-\vec{q})\,\delta(p_0-|q_0|)\left|\langle 0\big|J_\mu(0)\big|X(p)\rangle\right|^2$$

Мы уже определяли физическую спектральную плотность

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)}\delta^{(4)}(q-p)\,\theta(p_0)\left|\langle 0\big|J_\mu(0)\big|X(p)\rangle\right|^2$$

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Итак, мы имеем

$$\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$$

Почему в кореляторе T-произведение?

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Мы теперь в состоянии сказать каков смысл изучения T-произведений в корреляторах:

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Мы теперь в состоянии сказать каков смысл изучения T-произведений в корреляторах: **их мнимые части релятивистски-инвариантны, зависят только от q^2 и, тем не менее, связаны с реальными частицами!**

Почему в кореляторе T-произведение?

So, we have

$$\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)$$

and now we are able to say why we use T-products in correlators: **their imaginary parts are relativistic-invariant, depend only on q^2, and, nevertheless, are related to real particles**:

$$\rho(s) \;=\; \frac{s\,\sigma(s)}{16\,\pi^3\,\alpha^2} \;=\; \frac{1}{12\,\pi^2}\,R(s)$$

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$$R(s) \;=\; \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Dispersion relation in vector channel

We have $\Pi(Q^2) = \Pi(0) - Q^2 \int_0^\infty \frac{\rho(s)}{s(s+Q^2)}$, where $\rho(s) = \frac{1}{12\,\pi^2}\,\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$ is experimentally measurable quantity:

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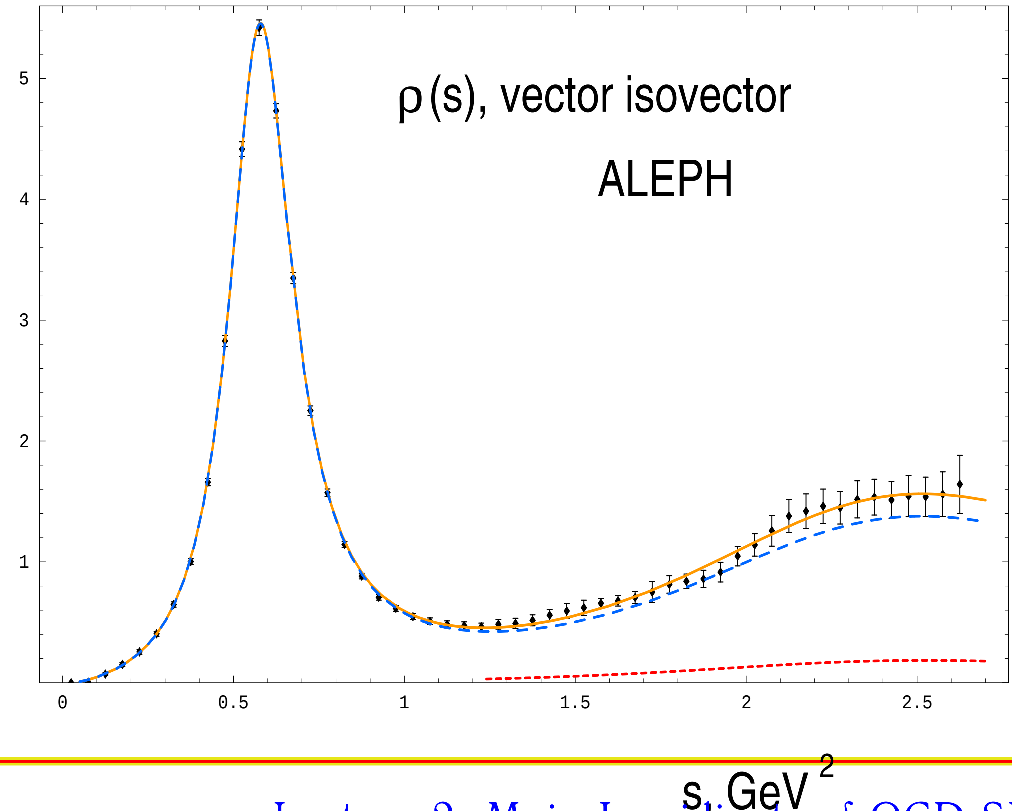

Quark-hadron duality

Global duality relation (theorem):

$$\int_0^\infty \left[\rho_{\rm pert}(s) - \rho_{\rm had}(s)\right] ds = 0$$

Quark-hadron duality

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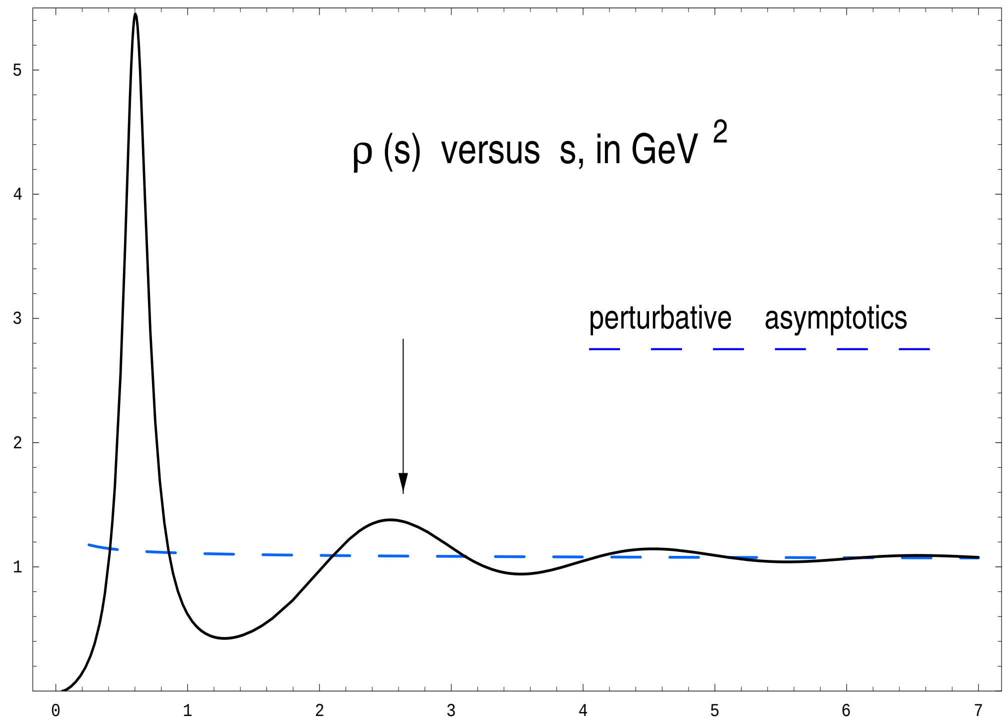
Local duality relation (hypothesis):

$$\int_{s_1}^{s_2} \left[\rho_{\text{pert}}(s) - \rho_{\text{had}}(s)\right] ds = 0$$

Quark-hadron duality

$$\int_{s_1}^{s_2} \rho_{\text{pert}}(s)ds = \int_{s_1}^{s_2} \rho_{\text{had}}(s)ds$$



Borel Transform and Condensates

Borel Transform

Borel transform is defined as

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$\frac{1}{s+Q^2}$	$\Rightarrow$	$\frac{1}{M^2}\,e^{-s/M^2}$

QCD ⇔ *Quantum Mechanics*

Сравнение КХД-переменных с квантовомеханическими аналогами:

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$\Phi(M^2)$	$\Leftrightarrow$	$M(\mu)$

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The question is: what are the contributions analogous to the ω^2/μ^2 corrections?

Pert. vs Non-Pert. contributions in QCD

The standard idea about the QCD potential $V(r)$ is that it consists of a Coulomb-like part and a nonperturbative $\sim r$ long distance "confining" part:

$$V(r) = V^{\text{Coulomb}}(r) + V^{\text{conf}}(r)\,.$$

The magnitude of the "Coulomb" effects is determined by the QCD running coupling constant

$$\alpha_s(Q^2) = \frac{4\pi}{9\log(Q^2/\Lambda^2)} + \ldots$$

Our problem now is to take into account the effects due to the long-range nonperturbative part of the QCD potential.

Pert. vs Non-Pert. contributions in QCD

Study of the QM oscillator has demonstrated that $M(\mu)$ in the presence of the potential is different from free $M_0(\mu)$.

This difference vanishes at short distances and one can calculate exact $M(\mu)$ perturbatively, expanding in powers of the oscillator potential.

In QCD the confining potential $V^{\rm conf}(r)$ is not even known.

Pert. vs Non-Pert. contributions in QCD

Possible way out is to proceed as follows:

- to construct perturbation expansion in terms of quark and gluon propagators;
- to postulate that quark and gluon propagators are modified by the long-range confinement part of the QCD potential;
- but the modification is soft in a sense that at short distances the difference between exact and perturbative (free-field) propagators vanishes.

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Then we write the exact propagator $\mathcal{D}^{exact}(x)$ as a vacuum average of a T-product of fields in the exact vacuum Ω

$$\mathcal{D}^{\text{exact}}(x) = \langle\Omega|\,T(\varphi(x)\varphi(0))|\Omega\rangle\,.$$

Pert. vs Non-Pert. contributions in QCD

According to the Wick theorem, one can write the T-product as the sum

$$T(\varphi(x)\varphi(0)) = \underbrace{\varphi(x)\varphi(0)} + :\varphi(x)\varphi(0):$$

of the "pairing" and the "normal" product.
The "pairing" is just the expectation value of the T-product over the perturbative vacuum

$$\underbrace{\varphi(x)\varphi(0)} = \langle 0|\, T(\varphi(x)\varphi(0))|0\rangle.$$

i.e., the perturbative propagator. By this definition, the normal product $:\varphi(x)\varphi(0):$ vanishes if averaged over the perturbative vacuum: $\langle 0|\ :\varphi(x)\varphi(0):|0\rangle = 0$.

Pert. vs Non-Pert. contributions in QCD

Thus, our assumption that $\mathcal{D}^{\text{exact}}(x) \neq \mathcal{D}^{\text{pert}}(x)$ is equivalent to the statement

$$\langle\, \varphi(x)\varphi(0)\rangle \equiv \langle\Omega|\ :\varphi(x)\varphi(0):|\Omega\rangle \neq 0,$$

which is the starting point to calculating power corrections in QCD.

Condensates in QCD

In the oscillator case the analog of $\langle\varphi(x)\varphi(0)\rangle$ is the difference between the exact Green function $G^{\rm osc}(\tau/i) = \dfrac{m\omega}{2\pi\sinh(\omega\tau)}$ and the free Green function $G^{\rm free}(\tau/i) = \dfrac{m}{2\pi\tau}$, where $\tau \equiv 1/\mu$ is just the imaginary time variable. Note, that both oscillator and free Green function are singular for $\tau \to 0$, but the difference

$$G^{\rm osc}(\tau/i) - G^{\rm free}(\tau/i) = \frac{m}{2\pi}\left[-\frac{1}{6}\,\omega^2\tau + \frac{7}{360}\,\omega^4\tau^3 - \ldots\right]$$

is regular at that point, and one can even expand the difference into the Taylor series in $(\omega\tau)^2$.

Condensates in QCD

In the QCD SR approach it is also assumed that the confinement effects are sufficiently soft to allow for the Taylor expansion of $\langle\, \varphi(x)\varphi(0)\rangle$ at $x = 0$:

$$\langle\varphi(0)\varphi(x)\rangle = \langle\varphi\varphi\rangle + x^{\mu}\langle\varphi\partial_{\mu}\varphi\rangle + \frac{x^{\mu_1}x^{\mu_2}}{2}\langle\varphi\partial_{\mu_1}\partial_{\mu_2}\varphi\rangle + \ldots$$

This is, in fact, the expansion of the nonlocal object $\langle\, \varphi(0)\varphi(x)\rangle$ over the vacuum matrix elements of the local composite operators. Not all operators really contribute to the expansion above ($\langle\varphi\partial_{\mu}\varphi\rangle = 0$, $\langle\varphi\partial_{\mu}\partial_{\nu}\varphi\rangle \sim g_{\mu\nu}\langle\varphi\partial^2\varphi\rangle$, etc.), so that finally one arrives at the expansion in x^2:

$$\langle\varphi(0)\varphi(x)\rangle = \sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n \frac{1}{n!(n+1)!}\,\langle\varphi(\partial^2)^n\varphi\rangle\,.$$

Condensates in QCD

Thus, the modification of the propagator by the nonperturbative effects is now parametrized by the matrix elements of the composite operators like $\langle\varphi(\partial^2)^n\varphi\rangle$. The examples in QCD are

- $\langle\bar{q}q\rangle$ referred to as the quark condensate;
- $\langle\bar{q}D^2q\rangle$, characterizing the average virtuality of the vacuum quarks;
- the gluon condensate $\langle G^a_{\mu\nu}G^a_{\mu\nu}\rangle$, etc.

Here $D_\mu \equiv \partial_\mu - igA_\mu$ is the covariant derivative and $G_{\mu\nu} = (i/g)[D_\mu, D_\nu]$ is the gluonic field strength.
Note, that only gauge invariant composite operators should appear in QCD, *i.e.* each ∂_μ must be accompanied by the relevant A_μ.

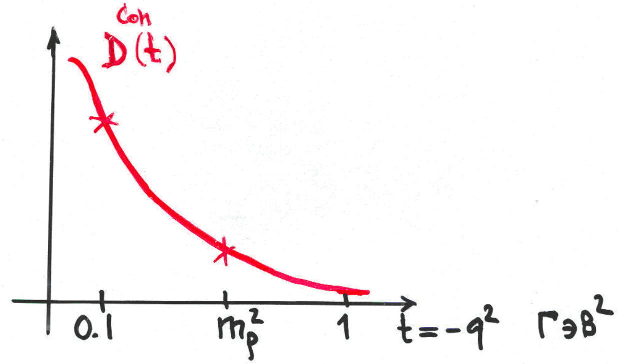
Конденсаты: распределение по импульсу.

Почему $V^{conf}(r)$ не портит стандартную ТВ?

$$\langle \varphi(0)\varphi(x)\rangle = \int e^{iqx} D^{con}(q)\, d^4q =$$

$$\sum_{n=0} \left(\frac{x^2}{4}\right)^n \frac{1}{n!\,(n+1)!} \int (-q^2)^n D^{con}(q)\, d^4q$$

$D^{con}(q)$ убывает БЫСТРЕЕ $\forall$ степени $(q^2)^{-m}$



$$D^{Exact}(q^2) = \frac{1}{q^2} + D^{con}(q^2)$$

↑ локализовано при $q^2 \sim 0$

$$D^{con} \sim e^{q^2/\Lambda^2}$$

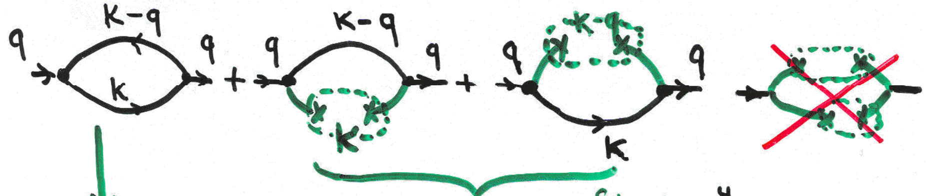
$$\sim K_1\left(r\sqrt{\Lambda^2+t}\right)\frac{1}{\sqrt{\Lambda^2+t}}$$

Совместимы с решеточными численными оценками.

Конденсаты в корреляторе

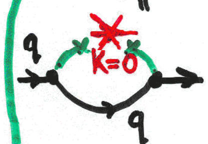
подход к операторному разложен. (OPE)

$K(q)$:



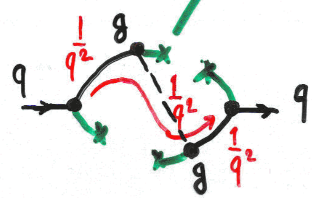
$$\int \frac{d^4k}{k^2(k-q)^2} \sim \log\left(-q^2/\mu^2\right) + c$$

$$I(q^2) = 2\int \frac{D^{con}(k^2)\, d^4k}{(q-k)^2} \sim 2\,\frac{\langle \varphi(0)\varphi(0)\rangle}{q^2}$$



$$K(q) = C_0 \cdot \log\left(-q^2/\mu^2\right) + C_1 \cdot \frac{\langle \varphi\varphi\rangle}{q^2} + C_3 \cdot g^2 \frac{\langle \varphi\varphi\rangle^2}{q^6} + \dots$$

В следующих порядках ТВ



ОПЕРАТОРНОЕ РАЗЛОЖЕНИЕ

Координатное представление

$$D^{free}(x,0)\,\langle\varphi(0)\,\varphi(x)\rangle = \frac{\langle\varphi\varphi\rangle}{x^2} + \frac{1}{x^2}\cdot\frac{x^2}{8}\langle\varphi\,\partial^2\varphi\rangle + O(x^2) + .$$

– ЧАСТНЫМ СЛУЧАЕМ ОПЕРАТОРНОГО РАЗЛОЖЕН
(OPE)

$$T\big(j(x)\,j(0)\big) = \sum_{n=0} C_n(x^2)\;\hat{O}_n(0)$$

$$C_n(x^2) \sim \frac{1}{(x^2)^{N_0-n}} \quad \text{и} \quad \hat{O}_n(0) \sim (\mu^2)^{d_0+n}$$

$$\text{Фурье}_q[C_n]:\ \left(\frac{\tilde{\mu}^2}{q^2}\right)^{[D/2-N_0+n]}$$

$$I(q^2) = \int \frac{D^{con}(k^2)\, d^4k}{(q-k)^2} = \frac{\langle \varphi(0)\varphi(0)\rangle}{q^2} + \text{"быстро убывающий вклад"}$$

$$\int \left(\frac{1}{k^2} - \frac{1}{q^2}\right) D^{con}(k^2)\, \theta(k^2 > q^2)\, d^4k =$$

модель: $D \sim e^{-q^2/\lambda^2}$

$$\sim - \frac{\langle \varphi(0)\varphi(0)\rangle}{q^2} \exp\{-q^2/\lambda^2\}$$

$$\lambda^2 = \frac{\langle \varphi \partial^2 \varphi\rangle}{\langle \varphi\varphi\rangle} = \langle k_\varphi^2\rangle$$

Локальный предел "работает", когда

$$\left(\lambda^2 \equiv \langle k_\varphi^2\rangle\right) \ll \left(\text{характерного } q^2, \sim E_0 \text{ или } m_h^2 \text{ или } \lesssim s_0\right)$$

Замечание о КХД

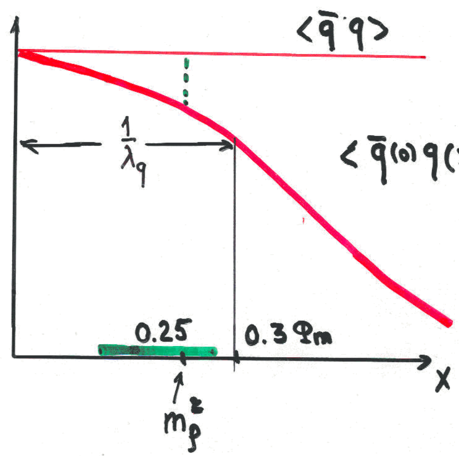
(1) $\langle \bar{q} q \rangle$, $m_q \langle \bar{q} q \rangle$ – кварковые конденсаты

(2) $\langle \bar{q} D^2 q \rangle$, $\lambda_q^2 = \langle \bar{q} D^2 q \rangle / \langle \bar{q} q \rangle$

(3) $\langle G^a_{\mu\nu} G^a_{\mu\nu} \rangle$ – глюонный конденсат

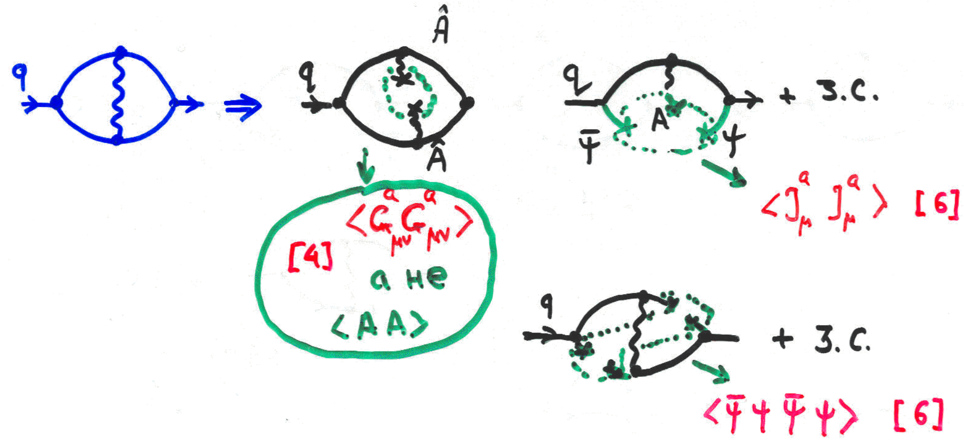
$$D_\mu = \partial_\mu - i g A^a_\mu t_a \, ; \qquad G^a_{\mu\nu} t_a = \frac{i}{g} [D_\mu, D_\nu]$$

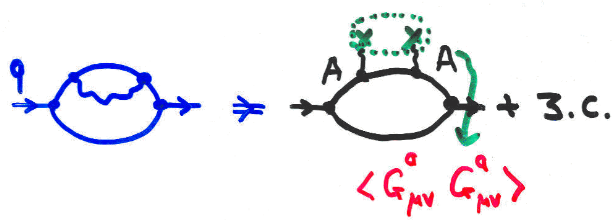
Когда может быть достаточно (1) – (3) параметров КХД вакуума?

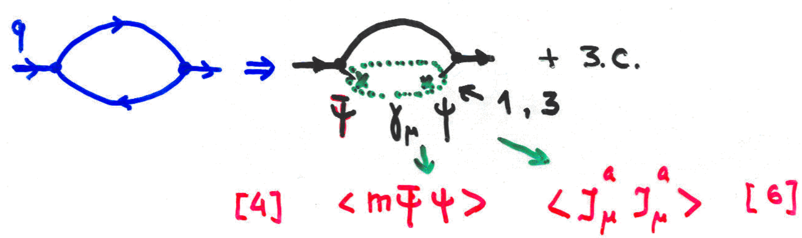


$$\langle \bar{q}(0) q(x) \rangle = \langle \bar{q} q \rangle \cdot \left(1 - \frac{x^2}{8} \lambda_q^2 \ldots \right)$$

КХД конденсатные диаграммы

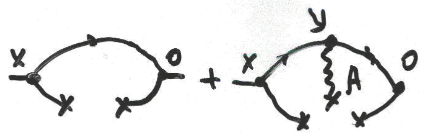






Конденсаты в калибровочной теории

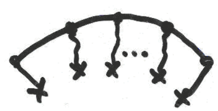
Не $x^\mu \langle \bar q \partial_\mu q \rangle$ $\quad x^\mu \langle \bar q \hat D_\mu q \rangle$?



$\to \langle \bar q \; ig A_\mu(0) q \rangle \int S(-y)\gamma^\mu S(y-x)\, d^D y = -x^\mu S(-x)$

$\langle \bar q \partial_\mu q \rangle$
$x^\mu S(-x)$

$S(-x)\, x^\mu \langle \bar q D_\mu q \rangle$



$$E(0,x) = P\exp\left\{ig \int_C \hat A_\mu(\tau)\, d\tau^\mu\right\}$$

$\langle \bar q(0)\, q(x) \rangle \to \langle \bar q(0)\, E(0,x)\, q(x) \rangle$

калибровочно-инвариантно

$E(0,x) = P\exp\left\{ig\, x^\mu \int_0^1 A_\mu(t\cdot x)\, dt\right\}$; $C =$ 

$x^\mu A_\mu(x) = 0$ калибровка { Фока-Швингера / Фикс. точки

$E(0,x) = 1$

$$\langle \bar q(0) E(0,x) q(x)\rangle\Big|_{\Phi T} = \langle \bar q(0) q(x)\rangle = \sum_{n=0} \frac{x_{\mu_1}\ldots x_{\mu_n}}{n!}\langle \bar q D_{\mu_1}\ldots D_{\mu_n} q\rangle$$

$$= \langle \bar q q\rangle + \frac{x^2}{8}\langle \bar q D^2 q\rangle + \ldots$$

$$\langle \bar q D^2 q\rangle = \frac{1}{2}\langle \bar q g (G^{\mu\rho}\sigma_{\mu\rho}) q\rangle - m_q^2 \langle \bar q q\rangle$$

$$\langle \bar q(0) q(x)\rangle = \langle \bar q q\rangle \left(1 - \frac{\tilde x^2}{4}\frac{1}{2}\frac{\langle \bar q D^2 q\rangle}{\langle \bar q q\rangle} + \ldots\right)$$

$$\lambda_q^2 \simeq \frac{\langle \bar q g (G^{\mu\rho}\sigma_{\mu\rho}) q\rangle}{2\langle \bar q q\rangle}$$

$\approx 0.4 \cdot$ ГэВ^2

Глюонный конденсат

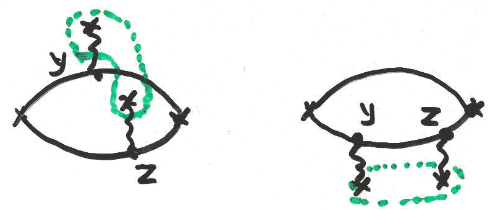
$$A_\mu^a(z)\Big|_{\Phi T} = z^\nu \int_0^1 G^b_{\nu\mu}(zt)\, t\, dt\, \tilde E^{ba}(z,0)$$

$\tilde E^{ba}(z,0) = \delta^{ba}$

A^a_μ ↑ поле; $G_{\nu\mu}$ ↑ напряженность

$$z^\nu \sum_{n=0}^{\infty} \frac{z^{\mu_1}\ldots z^{\mu_n}}{n!\,(n+2)} G_{\nu\mu;\mu_1\ldots\mu_n} = \frac{z^\nu}{2} G^a_{\nu\mu}(0) +$$

$$\tilde D^{ac}_{\mu_1} = \partial_{\mu_1} + g A^b f^{abc}$$



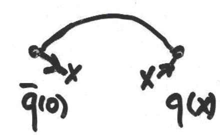
$$\langle : A^a_\mu(y)\, A^b_\nu(z) : \rangle = \frac{\delta^{ab}}{(N_c^2-1)}\, y^\rho z^\sigma \langle : G^a_{\mu\rho}\, G^a_{\nu\sigma} : \rangle + \ldots$$

$$\frac{\delta^{ab}}{8}\, y^\rho z^\sigma \left(g_{\mu\nu}\, g_{\rho\sigma} - g_{\rho\nu}\, g_{\mu\sigma} \right) \langle G^a_{\rho'\sigma'}\, G^a_{\rho'\sigma'} \rangle$$

$$\downarrow$$

$$\langle G^2 \rangle$$

Кварковый конденсат



$$\langle \bar{q}\, \gamma_\mu\, \hat{S}(-x)\, \gamma_\nu\, q \rangle$$

$$\downarrow \; F(q)$$

$$\langle \bar{q}\, \gamma^\mu \frac{m-\hat{q}}{q^2-m^2}\, \gamma^\nu q \rangle \sim \frac{\langle m\bar{q}q\rangle g^{\mu\nu}}{q^2}$$

?

поперечность ?

$$\langle \bar{q}(0)\, \gamma_\mu \left(\frac{2i\,\hat{x}}{x^4} + \frac{m}{x^2} \right) \gamma_\nu\, q(x) \rangle$$