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Взаимодействующее поле Рариты-Швингера и спин-четность его компонент

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1 Introduction

The covariant description of the spin 3/2 particles is usually based on the Rarita-Schwinger formalism [1] where the main object is the spin-vector field Ψ^μ.

The most general free lagrangian:

$$\begin{aligned}
\mathcal{L} &= \overline{\Psi}^\mu \Lambda^{\mu\nu} \Psi^\nu, \\
\Lambda^{\mu\nu} &= (\hat{p} - M) g^{\mu\nu} + A(\gamma^\mu p^\nu + \gamma^\nu p^\mu) + \frac{1}{2}(3A^2 + 2A + 1)\gamma^\mu \hat{p} \gamma^\nu + \\
M &\ \ (3A^2 + 3A + 1)\gamma^\mu \gamma^\nu. \qquad (1)
\end{aligned}$$

Here A is an arbitrary parameter, $p_\mu = i\partial_\mu$.

This lagrangian is invariant under the point transformation:

$$\Psi^\mu \to \Psi'^\mu = (g^{\mu\nu} + \alpha\gamma^\mu\gamma^\nu)\Psi^\nu, \qquad A \to A' = \frac{A - 2\alpha}{1 + 4\alpha},$$

with parameter $\alpha \neq -1/4$.

The lagrangian (1) leads to the following equations of motion:

$$\Lambda^{\mu\nu}\Psi^\nu = 0. \qquad (2)$$

The free propagator of the Rarita-Schwinger field in a momentum space obeys the equation:

$$\Lambda^{\mu\nu} G_0^{\nu\rho} = g^{\mu\rho}. \qquad (3)$$

The expression for the free propagator $G_0^{\mu\nu}$ is well known and we do not present it here.

As concerned for the dressed propagator, its construction is a more complicated issue and its total expression is unknown up to now.

Here we derive an analytical expression for the interacting Rarita-Schwinger field's propagator with accounting all spin components and discuss its properties. It turned out that the spin 1/2 part of the dressed propagator has rather compact form, and a crucial point for its deriving is the choosing of a suitable basis. Short variant of this paper was published (A.E. Kaloshin, V.P. Lomov. Mod.Phys.Lett. A19 (2004) 135).

2 Dressed propagator of the Rarita-Schwinger field

The Dyson-Schwinger equation for the propagator

$$G^{\mu\nu} = G_0^{\mu\nu} + G^{\mu\alpha} J^{\alpha\beta} G_0^{\beta\nu}. \quad (4)$$

Here $G_0^{\mu\nu}$ and $G^{\mu\nu}$ are the free and full propagators respectively, $J^{\mu\nu}$ is a self-energy contribution. The equation may be rewritten for inverse propagators as

$$(G^{-1})^{\mu\nu} = (G_0^{-1})^{\mu\nu} - J^{\mu\nu}. \quad (5)$$

If we consider the self-energy $J^{\mu\nu}$ as a known value ("rainbow approximation", see *e.g.* recent review[16]), than the problem is reduced to reversing of relation (5). It is useful to have a basis for both propagators and self-energy.

1. The most natural basis for the spin-tensor $S^{\mu\nu}(p)$ decomposition is the γ-matrix one:

$$\begin{aligned} S^{\mu\nu}(p) =& g^{\mu\nu} \cdot s_1 + p^\mu p^\nu \cdot s_2 + \\ &+ \hat{p} p^\mu p^\nu \cdot s_3 + \hat{p} g^{\mu\nu} \cdot s_4 + p^\mu \gamma^\nu \cdot s_5 + \gamma^\mu p^\nu \cdot s_6 + \\ &+ \sigma^{\mu\nu} \cdot s_7 + \sigma^{\mu\lambda} p^\lambda p^\nu \cdot s_8 + \sigma^{\nu\lambda} p^\lambda p^\mu \cdot s_9 + \gamma^\lambda \gamma^5 \imath \varepsilon^{\lambda\mu\nu\rho} p^\rho \cdot s_{10}. \end{aligned} \quad (6)$$

   Here $S^{\mu\nu}$ is an arbitrary spin-tensor depending on the momentum p, $s_i(p^2)$ are the Lorentz invariant coefficients, and $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Altogether there are ten independent components in the decomposition of $S^{\mu\nu}(p)$.

   It is known that the γ-matrix decomposition is complete, the coefficients s_i are free of kinematical singularities and constraints, and their calculation is rather simple. However this basis is inconvenient at multiplication and reversing of the spin-tensor $S^{\mu\nu}(p)$ because the basis elements are not orthogonal to each other. As a result the reversing of the spin-tensor $S^{\mu\nu}(p)$ leads to a system of 10 equations for the coefficients.

2. There is another basis used in consideration of the dressed propagator [4, 9, 13] $G^{\mu\nu}$. It is constructed from the following set of operators [12, 13, 17]

$$
\begin{aligned}
(\mathcal{P}^{3/2})^{\mu\nu} =& g^{\mu\nu} - \frac{2}{3}\frac{p^\mu p^\nu}{p^2} - \frac{1}{3}\gamma^\mu\gamma^\nu + \frac{1}{3p^2}(\gamma^\mu p^\nu - \gamma^\nu p^\mu)\hat{p},\\
(\mathcal{P}^{1/2}_{11})^{\mu\nu} =& \frac{1}{3}\gamma^\mu\gamma^\nu - \frac{1}{3}\frac{p^\mu p^\nu}{p^2} - \frac{1}{3p^2}(\gamma^\mu p^\nu - \gamma^\nu p^\mu)\hat{p},\\
(\mathcal{P}^{1/2}_{22})^{\mu\nu} =& \frac{p^\mu p^\nu}{p^2},\\
(\mathcal{P}^{1/2}_{21})^{\mu\nu} =& \sqrt{\frac{3}{p^2}}\cdot\frac{1}{3p^2}(-p^\mu + \gamma^\mu\hat{p})\hat{p}p^\nu,\\
(\mathcal{P}^{1/2}_{12})^{\mu\nu} =& \sqrt{\frac{3}{p^2}}\cdot\frac{1}{3p^2}p^\mu(-p^\nu + \gamma^\nu\hat{p})\hat{p}.
\end{aligned}
\tag{7}
$$

Let us rewrite the operators (7) to make their properties more obvious:

$$
\begin{aligned}
(\mathcal{P}^{3/2})^{\mu\nu} =& g^{\mu\nu} - (\mathcal{P}^{1/2}_{11})^{\mu\nu} - (\mathcal{P}^{1/2}_{22})^{\mu\nu},\\
(\mathcal{P}^{1/2}_{11})^{\mu\nu} =& 3\pi^\mu\pi^\nu,\\
(\mathcal{P}^{1/2}_{22})^{\mu\nu} =& \frac{p^\mu p^\nu}{p^2},\\
(\mathcal{P}^{1/2}_{21})^{\mu\nu} =& \sqrt{\frac{3}{p^2}}\cdot\pi^\mu p^\nu,\\
(\mathcal{P}^{1/2}_{12})^{\mu\nu} =& \sqrt{\frac{3}{p^2}}\cdot p^\mu\pi^\nu.
\end{aligned}
\tag{8}
$$

Here we introduced the vector

$$
\pi^\mu = \frac{1}{3p^2}(-p^\mu + \gamma^\mu\hat{p})\hat{p}
\tag{9}
$$

with the following properties:

$$
(\pi p) = 0, \quad (\gamma\pi) = (\pi\gamma) = 1, \quad (\pi\pi) = \frac{1}{3}, \quad \hat{p}\pi^\mu = -\pi^\mu\hat{p}.
\tag{10}
$$

Here $\mathcal{P}^{3/2}$,$\mathcal{P}^{1/2}_{11}$,$\mathcal{P}^{1/2}_{22}$ are the projection operators while $\mathcal{P}^{1/2}_{21}$,$\mathcal{P}^{1/2}_{12}$ are nilpotent. As for their physical meaning, it is clear that $\mathcal{P}^{3/2}$ corresponds to spin 3/2. The remaining operators should describe two spin 1/2 representations and transitions between them.

The set of operators (7) can be used to decompose the considered spin-tensor as following [4, 9]:

$$\begin{aligned} S^{\mu\nu}(p) &= (S_1 + S_2\hat{p})(\mathcal{P}^{3/2})^{\mu\nu} + (S_3 + S_4\hat{p})(\mathcal{P}^{1/2}_{11})^{\mu\nu} + \\ &\quad (S_5 + S_6\hat{p})(\mathcal{P}^{1/2}_{22})^{\mu\nu} + (S_7 + S_8\hat{p})(\mathcal{P}^{1/2}_{21})^{\mu\nu} + \\ &\quad (S_9 + S_{10}\hat{p})(\mathcal{P}^{1/2}_{12})^{\mu\nu}. \end{aligned} \tag{11}$$

Let us call this basis as $\hat{p}$-basis. It is more convenient at multiplication since the spin 3/2 components $\mathcal{P}^{3/2}$ have been separated from spin 1/2 ones. However, the spin 1/2 components as before are not orthogonal between themselves and we come to a system of 8 equations when inverting the (5). Another feature of decomposition (11) is existence of the poles $1/p^2$ in different terms. So to avoid this unphysical singularity, we should impose some constraints on the coefficients at zero point.

3. Let us construct the basis which is the most convenient at multiplication of spin-tensors. This basis is built from the operators (7) and the projection operators $\Lambda^{\pm}$

$$\Lambda^{\pm} = \frac{\sqrt{p^2} \pm \hat{p}}{2\sqrt{p^2}}, \tag{12}$$

where we assume $p^2 > 0$. Ten elements of this basis look as

$$\begin{aligned} \mathcal{P}_1 &= \Lambda^+\mathcal{P}^{3/2},\ \mathcal{P}_2 = \Lambda^-\mathcal{P}^{3/2}, \\ \mathcal{P}_3 &= \Lambda^+\mathcal{P}^{1/2}_{11},\ \mathcal{P}_4 = \Lambda^-\mathcal{P}^{1/2}_{11}, \\ \mathcal{P}_5 &= \Lambda^+\mathcal{P}^{1/2}_{22},\ \mathcal{P}_6 = \Lambda^-\mathcal{P}^{1/2}_{22}, \\ \mathcal{P}_7 &= \Lambda^+\mathcal{P}^{1/2}_{21},\ \mathcal{P}_8 = \Lambda^-\mathcal{P}^{1/2}_{21}, \\ \mathcal{P}_9 &= \Lambda^+\mathcal{P}^{1/2}_{12},\ \mathcal{P}_{10} = \Lambda^-\mathcal{P}^{1/2}_{12}. \end{aligned} \tag{13}$$

where tensor indices are omitted. We will call (13) as the Λ-basis.

Decomposition of a spin-tensor in this basis has the following form:

$$S^{\mu\nu}(p) = \sum_{i=1}^{10} \mathcal{P}_i^{\mu\nu} \bar{S}_i(p^2). \tag{14}$$

The Λ-basis has very simple multiplicative properties which are represented in the Table 1.

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$	$\mathcal{P}_8$	$\mathcal{P}_9$	$\mathcal{P}_{10}$
$\mathcal{P}_1$	$\mathcal{P}_1$	0	0	0	0	0	0	0	0	0
$\mathcal{P}_2$	0	$\mathcal{P}_2$	0	0	0	0	0	0	0	0
$\mathcal{P}_3$	0	0	$\mathcal{P}_3$	0	0	0	$\mathcal{P}_7$	0	0	0
$\mathcal{P}_4$	0	0	0	$\mathcal{P}_4$	0	0	0	$\mathcal{P}_8$	0	0
$\mathcal{P}_5$	0	0	0	0	$\mathcal{P}_5$	0	0	0	$\mathcal{P}_9$	0
$\mathcal{P}_6$	0	0	0	0	0	$\mathcal{P}_6$	0	0	0	$\mathcal{P}_{10}$
$\mathcal{P}_7$	0	0	0	0	0	$\mathcal{P}_7$	0	0	0	$\mathcal{P}_3$
$\mathcal{P}_8$	0	0	0	0	$\mathcal{P}_8$	0	0	0	$\mathcal{P}_4$	0
$\mathcal{P}_9$	0	0	0	$\mathcal{P}_9$	0	0	0	$\mathcal{P}_5$	0	0
$\mathcal{P}_{10}$	0	0	$\mathcal{P}_{10}$	0	0	0	$\mathcal{P}_6$	0	0	0

Table 1: Properties of the Λ-basis at multiplication.

The first six basis elements are projection operators, while the remaining four elements are nilpotent.

Now we can return to the Dyson-Schwinger equation (5). Let us denote the inverse dressed propagator $(G^{-1})^{\mu\nu}$ and free one $(G_0^{-1})^{\mu\nu}$ by $S^{\mu\nu}$ and $S_0^{\mu\nu}$ respectively. Decomposing the $S^{\mu\nu}$, $S_0^{\mu\nu}$ and $J^{\mu\nu}$ in Λ-basis according to (14) we reduce the equation (5) to set of equations for the scalar coefficients

$$\bar{S}_i(p^2) = \bar{S}_{0i}(p^2) + \bar{J}_i(p^2)$$

So the values $\bar{S}_i$ are defined by the bare propagator and the self-energy and may be considered as known.

The dressed propagator also can be found in this form

$$G^{\mu\nu} = \sum_{i=1}^{10} \mathcal{P}_i^{\mu\nu} \cdot \bar{G}_i(p^2) \tag{15}$$

The existing 6 projection operators take part in the decomposition of $g^{\mu\nu}$:

$$g^{\mu\nu} = \sum_{i=1}^{6} \mathcal{P}_i^{\mu\nu}. \tag{16}$$

Now solving the equation

$$G^{\mu\nu} S^{\nu\lambda} = g^{\mu\lambda}$$

in Λ-basis, we obtain a set of equations for the scalar coefficients $\bar{G}_i$, which are easy to solve:

$$\begin{aligned}
&\bar{G}_1 = \frac{1}{\bar{S}_1}, \quad \bar{G}_2 = \frac{1}{\bar{S}_2}, \\
&\bar{G}_3 = \frac{\bar{S}_6}{\Delta_1}, \quad \bar{G}_4 = \frac{\bar{S}_5}{\Delta_2}, \quad \bar{G}_5 = \frac{\bar{S}_4}{\Delta_2}, \quad \bar{G}_6 = \frac{\bar{S}_3}{\Delta_1}, \\
&\bar{G}_7 = \frac{-\bar{S}_7}{\Delta_1}, \quad \bar{G}_8 = \frac{-\bar{S}_8}{\Delta_2}, \quad \bar{G}_9 = \frac{-\bar{S}_9}{\Delta_2}, \quad \bar{G}_{10} = \frac{-\bar{S}_{10}}{\Delta_1},
\end{aligned} \tag{17}$$

where

$$\Delta_1 = \bar{S}_3\bar{S}_6 - \bar{S}_7\bar{S}_{10}, \qquad \Delta_2 = \bar{S}_4\bar{S}_5 - \bar{S}_8\bar{S}_9. \tag{18}$$

3 Joint dressing of Dirac fermions

The answer has rather unusual structure, so we would like to clear its physical meaning before renormalization. First of all it's useful to consider the dressing if Dirac fermions with aim to find the close analogy for Rarita-Schwinger field case.

3.1 Dressing of single Dirac fermion

The dressed fermion G(p) propagator is solution of the Dyson-Schwinger equation

$$G(p) = G_0 + G\Sigma G_0. \tag{19}$$

Let us use the basis of the projection operators but with new notatons to stress an analogy with the Rarita-Schwinger field.

$$\mathcal{P}_1 = \Lambda^+, \qquad \mathcal{P}_2 = \Lambda^-. \tag{20}$$

Decomposition of any matrix, depending on p :

$$S(p) = \sum_{M=1}^{2} \mathcal{P}_M \bar{S}^M. \tag{21}$$

Dyson-Schwinger equation in this basis takes the form:

$$\bar{G}^M = \bar{G}_0^M + \bar{G}^M \bar{\Sigma}^M \bar{G}_0^M, \qquad M = 1, 2, \tag{22}$$

and its solution is:

$$(\bar{G}^M)^{-1} = (\bar{G}_0^M)^{-1} - \bar{\Sigma}^M. \tag{23}$$

In more detail:

$$(\bar{G}^{M=1})^{-1} = (\bar{G}_0^{M=1})^{-1} - \bar{\Sigma}^{M=1} = -m_0 - A(p^2) + \sqrt{p^2}\big(1 - B(p^2)\big),$$
$$(\bar{G}^{M=2})^{-1} = (\bar{G}_0^{M=2})^{-1} - \bar{\Sigma}^{M=2} = -m_0 - A(p^2) - \sqrt{p^2}\big(1 - B(p^2)\big),$$

where A, B are the conventional components of the self-energy

$$\Sigma(p) = A(p^2) + \hat{p} B(p^2) = \Lambda^+ \Sigma^1 + \Lambda^- \Sigma^2,$$
$$\Sigma^1 = A + \sqrt{p^2} B, \qquad \Sigma^2 = A - \sqrt{p^2} B.$$

Standard procedure of renormalization consist in formal expansion $G^{-1}(p)$ in terms of $\hat{p} - m$ and choosing the renormalization constants to fulfill the condition

$$G^{-1}(p) = \hat{p} - m + o(\hat{p} - m).$$

With using of the projection operators basis one needs to renormalize the scalar functions G^M, depending on the argument $E = \sqrt{p^2}$.

Let us consider the $(\bar{G}^{\ 1})^{-1}$ component (recall that the bare contribution is $(G_0^1)^{-1} = -m_0 + \sqrt{p^2}$) and require its expansion in terms on $\big(\sqrt{p^2} - m\big)$ to be:

$$\left(\bar{G}^1\right)^{-1} = \sqrt{p^2} - m + o\big(\sqrt{p^2} - m\big)$$

As a result we have the dressed renormalized propagator $G(p)$, which coincides with the standard expression.

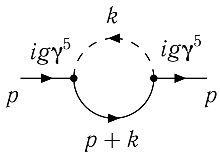
Let us look at the self-energy contribution $\Sigma(p)$. As an example we shall consider the dressing of barion resonance N' ($J^P = 1/2^{\pm}$) due to interaction with πN system. Interaction lagrangian is of the form

$$L_{int} = g\overline{\Psi}\,'(x)\gamma^5\Psi(x)\cdot\phi(x) + h.c. \quad \text{for} \quad N' = 1/2^+ \tag{24}$$

and

$$L_{int} = g\overline{\Psi}\,'(x)\Psi(x)\cdot\phi(x) + h.c. \quad \text{for} \quad N' = 1/2^-. \tag{25}$$

$\boxed{\frac{1}{2}^+ \leftrightarrow \frac{1}{2}^+}$



$$\Sigma(p) = ig^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^5\frac{1}{\hat{p}+\hat{k}-m_N}\gamma^5\frac{1}{k^2-m_\pi^2} = I\cdot A(p^2) + \hat{p}B(p^2) \tag{26}$$

Calculate the loop discontinuity through the Landau-Cutkosky rule:

$$\Delta A = -\frac{ig^2 m_N}{(2\pi)^2}I_0, \qquad \Delta B = \frac{ig^2}{(2\pi)^2}I_0\frac{p^2+m_N^2-m_\pi^2}{2p^2}. \tag{27}$$

Here I_0 is the base integral

$$I_0 = \int d^4k\,\delta\big(k^2-m_\pi^2\big)\,\delta\big((p+k)^2-m_N^2\big) = \frac{\pi}{2}\theta\big(p^2-(m_N+m_\pi)^2\big)\frac{q}{E},$$

where $\lambda\big(a,b,c\big) = \big(a-b-c\big)^2 - 4bc$, q is the momentum of πN pair in CMS.

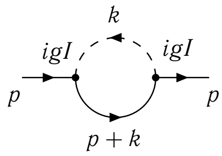
From the parity conservation one can see that in the transition $N'(1/2^+)\to N(1/2^+)+\pi(0^-)$ the πN pair has the orbital momentum $l=1$. But according to threshold quantum-mechanical theorems [19], the imaginary part of a loop should behave as q^{2l+1} at $q\to 0$, which does not correspond to (27).

Imaginary part of $\bar{\Sigma}^M$ component according to (27)

$$Im\ \bar{\Sigma}^1 = Im\ \big(A + \sqrt{p^2}B\big) \sim q^3,$$
$$Im\ \bar{\Sigma}^2 = Im\ \big(A - \sqrt{p^2}B\big) \sim q^1. \qquad (28)$$

One can see that ,,alive" component Σ^1 demonstrates the proper threshold behavior.

$\boxed{\frac{1}{2}^- \leftrightarrow \frac{1}{2}^-}$



$$\Sigma(p) = ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k} + \hat{p} - m_N} \cdot \frac{1}{k^2 - m_\pi^2} = IA(p^2) + \hat{p}B(p^2),$$
$$\Delta A = -i\frac{g^2 m_N}{(2\pi)^2} I_0, \qquad \Delta B = \frac{-ig^2}{(2\pi)^2} I_0 \frac{p^2 + m_N^2 - m_\pi^2}{2p^2}$$

Imaginary parts of $\bar{\Sigma}^{1,2}$ now demonstrate $l = 0$ behavior

$$Im\ \bar{\Sigma}^1 = -\frac{g^2 I_0}{4\sqrt{p^2}(2\pi)^2}\Big[\big(\sqrt{p^2} + m_N\big)^2 - m_\pi^2\Big] \sim q^1,$$
$$Im\ \bar{\Sigma}^2 = \frac{g^2 I_0}{4\sqrt{p^2}(2\pi)^2}\big(\sqrt{p^2} - m_N - m_\pi\big)\big(\sqrt{p^2} - m_N + m_\pi\big) \sim q^3.$$

The considered examples show that only an ,,alive" component $\bar{\Sigma}^1$, which has the pole $1/\big(\sqrt{p^2} - m\big)$ demonstrates the proper threshold behavior (i.e. the proper parity). Another component $\bar{\Sigma}^2$, which has pole of the form $1/\big(-\sqrt{p^2} - m\big)$ demonstrates the opposite parity.

3.2 Dressing of Dirac fermion with parity violation

Let us consider dressing of the Dirac fermion in case of parity violation. Such situation arises for instance in case of the t-quark dressing. In this case the Dyson-Schwinger equation has the previous form but the self-energy contribution Σ has the parity violating terms.

$$\Sigma(p) = A(p^2) + \hat{p}B(p^2) + \gamma^5 C(p^2) + \hat{p}\gamma^5 D(p^2). \tag{29}$$

It is convinient to use the following four operators as a basis:

$$\mathcal{P}_1 = \Lambda^+, \quad \mathcal{P}_2 = \Lambda^-, \quad \mathcal{P}_3 = \Lambda^+\gamma^5, \quad \mathcal{P}_4 = \Lambda^-\gamma^5. \tag{30}$$

$\mathcal{P}_{1,2}$ are projection operators, while $\mathcal{P}_{3,4}$ are nilpotent ones. The decomposition of any γ-матрицы, depending on p, now is of the form (compaire with (21))

$$S(p) = \sum_{M=1}^{4} \mathcal{P}^M \bar{S}^M. \tag{31}$$

This set of operators has simple multiplicative properties (see Table 2).

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
$\mathcal{P}_1$	$\mathcal{P}_1$	0	$\mathcal{P}_3$	0
$\mathcal{P}_2$	0	$\mathcal{P}_2$	0	$\mathcal{P}_4$
$\mathcal{P}_3$	0	$\mathcal{P}_3$	0	$\mathcal{P}_1$
$\mathcal{P}_4$	$\mathcal{P}_4$	0	$\mathcal{P}_2$	0

Table 2: Multiplicative properties of the operators (41)

Let us denote by $S(p)$ and $S_0(p)$ the dressed and bare inverse propagators respectively. The Dyson-Schwinger equation with accounting (42) reduces to

$$\bar{S}^M = \left(\bar{S}_0\right)^M - \bar{\Sigma}^M, \qquad M = 1, \ldots, 4$$

so the components $\bar{S}^M$ may be considered as known and we came to problem of reversing of known matrix $S(p)$:

$$\left(\sum_{M=1}^{4} \mathcal{P}_M \bar{G}^M\right)\left(\sum_{L=1}^{4} \mathcal{P}_L \bar{S}^L\right) = \mathcal{P}_1 + \mathcal{P}_2. \tag{32}$$

After multiplication we obtain set of equations in $\bar{G}^M$

$$\begin{aligned} \bar{G}^1\bar{S}^1 + \bar{G}^3\bar{S}^4 &= 1 \\ \bar{G}^2\bar{S}^2 + \bar{G}^4\bar{S}^3 &= 1 \\ \bar{G}^1\bar{S}^3 + \bar{G}^3\bar{S}^2 &= 0 \\ \bar{G}^4\bar{S}^1 + \bar{G}^2\bar{S}^4 &= 0, \end{aligned} \tag{33}$$

which are easy to solve. Answer is:

$$\bar{G}_1 = \frac{\bar{S}_2}{\Delta}, \quad \bar{G}_2 = \frac{\bar{S}_1}{\Delta}, \quad \bar{G}_3 = -\frac{\bar{S}_3}{\Delta}, \quad \bar{G}_4 = -\frac{\bar{S}_4}{\Delta}, \tag{34}$$

где $\Delta = \bar{S}_1\bar{S}_2 - \bar{S}_3\bar{S}_4$.

This example is similar to dressing of the Rarita–Schwinger field by its algebraical structure (compare Tables 1, 2) but it has too small degrees of freedom.

3.3 Joint dressing of two fermions of the same parity

Let we have two bare fermion states N',N'', which are dressing with presence of mutual transitions. Suppose that these states have the same parity and there is no parity violation in lagrangian. Now the Dyson-Schwinger equation will have the martix indexes

$$G_{ij} = \big(G_0\big)_{ij} + G_{ik}\Sigma_{kl}\big(G_0\big)_{lj}, \qquad i, j, k, l = 1, 2. \tag{35}$$

Every element here has not shown γ-matrix indexes.

With using of the projection operators basis (21), we will reduce the equation (40) to independent equations for components $\mathcal{P}_1$, $\mathcal{P}_2$:

$$\big(\bar{G}^M\big)_{ij} = \big(\bar{G}^M_0\big)_{ij} + \big(\bar{G}^M\big)_{ik}\big(\bar{\Sigma}^M\big)_{kl}\big(\bar{G}^M_0\big)_{lj}, \qquad M = 1, 2. \tag{36}$$

Let us rewrite the Eq. (36) in matrix form

$$\bar{G}^M = \bar{G}^M_0 + \bar{G}^M\bar{\Sigma}^M\bar{G}^M_0, \quad M = 1, 2, \tag{37}$$

and write down its solution

$$G^M = \left[\left(\bar{G}_0^M\right)^{-1} - \bar{\Sigma}^M\right]^{-1} = \begin{pmatrix} \left(\bar{G}_0^M\right)_{11}^{-1} - \bar{\Sigma}_{11}^M & -\bar{\Sigma}_{12}^M \\ -\bar{\Sigma}_{21}^M & \left(\bar{G}_0^M\right)_{22}^{-1} - \bar{\Sigma}_{22}^M \end{pmatrix}^{-1} =$$
$$= \frac{1}{\Delta^M} \begin{pmatrix} \left(\bar{G}_0^M\right)_{22}^{-1} - \bar{\Sigma}_{22}^M & -\bar{\Sigma}_{12}^M \\ -\bar{\Sigma}_{21}^M & \left(\bar{G}_0^M\right)_{11}^{-1} - \bar{\Sigma}_{11}^M \end{pmatrix},$$
$$\Delta^M = \left[\left(\bar{G}_0^M\right)_{11}^{-1} - \bar{\Sigma}_{11}^M\right]\left[\left(\bar{G}_0^M\right)_{22}^{-1} - \bar{\Sigma}_{22}^M\right] - \bar{\Sigma}_{12}^M \bar{\Sigma}_{21}^M. \tag{38}$$

Now let us calculate the loop contributions Σ_{ij} for considered in above πN intermediate state. For dressing of two states N', N'' of the same parity the self energy coinsides with (26), (28) except the coupling constants.

$$\Sigma_{ij} = i g_i g_j \int \frac{d^4k}{(2\pi)^4}\, \gamma^5 \frac{1}{\hat{p}+\hat{k}-m} \gamma^5 \frac{1}{k^2 - m_\pi^2} \qquad for\ N', N'' = 1/2^+,$$
$$\Sigma_{ij} = i g_i g_j \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{\hat{p}+\hat{k}-m} \frac{1}{k^2 - m_\pi^2} \qquad for\ N', N'' = 1/2^-. \tag{39}$$

The threshold behavior of imaginary parts for dressing of two fermions (39) is similar to case of a single fermion, considered in previous section.

3.4 Joint dressing of two fermions of different parity

Let us consider the nearest analogy to the Rarita-Schwinger field: the joint dressing of two fermions of different parity $1/2^\pm$.

Now the Dyson-Schwinger equation has matrix form

$$G_{ij} = \left(G_0\right)_{ij} + G_{ik}\Sigma_{kl}\left(G_0\right)_{lj}, \qquad i, j, k, l = 1, 2. \tag{40}$$

The basis contains four operators:

$$\mathcal{P}_1 = \Lambda^+, \quad \mathcal{P}_2 = \Lambda^-, \quad \mathcal{P}_3 = \Lambda^+\gamma^5, \quad \mathcal{P}_4 = \Lambda^-\gamma^5, \tag{41}$$

where $\mathcal{P}_{1,2}$ are projection operators and $\mathcal{P}_{3,4}$ are the nilpotent ones. Decomposition of any γ-matrix, depending on p, now is of the form

$$S(p) = \sum_{M=1}^{4} \mathcal{P}^M \bar{S}^M. \tag{42}$$

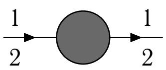
The Dyson-Schwinger equation (40) reduces to the matrix equations:

$$\begin{aligned} G_1 S_1 + G_3 S_4 &= E_2, \\ G_2 S_2 + G_4 S_3 &= E_2, \\ G_1 S_3 + G_3 S_2 &= 0, \\ G_4 S_1 + G_2 S_4 &= 0, \end{aligned} \tag{43}$$

where E_2 is the unit matrix 2×2. Solutions:

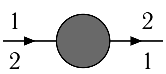
$$\begin{aligned} G_1 &= \Big[S_1 - S_3 \big(S_2\big)^{-1} S_4\Big]^{-1}, \\ G_2 &= \Big[S_2 - S_4 \big(S_1\big)^{-1} S_3\Big]^{-1}, \\ G_3 &= -\Big[S_1 - S_3 \big(S_2\big)^{-1} S_4\Big]^{-1} S_3 \big(S_2\big)^{-1}, \\ G_4 &= -\Big[S_2 - S_4 \big(S_1\big)^{-1} S_3\Big]^{-1} S_4 \big(S_1\big)^{-1}. \end{aligned} \tag{44}$$

Now let us concretize these general formulae. Suppose that we have two fermions of different parity, but there is no parity violation in lagrangian. It means that the diagonal loops contain only the I and $\hat{p}$



$\Sigma_{ii} \sim I,\ \hat{p}$,

while the non-diagonal ones have γ^5



$\Sigma_{ij} \sim \gamma^5,\ \hat{p}\gamma^5$ for $i \neq j$

Decomposition of inverse propagator in this basis has the form

$$S(p) \;=\; \mathcal{P}_1 \begin{pmatrix} -m_1 + E - \Sigma_{11}^{(1)} & 0 \\ 0 & -m_2 + E - \Sigma_{22}^{(1)} \end{pmatrix}$$

$$+\ \mathcal{P}_2\begin{pmatrix} -m_1-E-\Sigma^{(2)}_{11} & 0\\ 0 & -m_2-E-\Sigma^{(2)}_{22}\end{pmatrix}$$

$$+\ \mathcal{P}_3\begin{pmatrix} 0 & -\Sigma^{(3)}_{12}\\ -\Sigma^{(3)}_{21} & 0\end{pmatrix}+\mathcal{P}_4\begin{pmatrix} 0 & -\Sigma^{(4)}_{12}\\ -\Sigma^{(4)}_{21} & 0\end{pmatrix}.$$

Substituting all into solution (44), we have the dressed propagator

$$G=\Lambda^+\begin{pmatrix} \frac{-m_2-E-\Sigma^2_{22}}{\Delta_1} & 0\\ 0 & \frac{-m_1-E-\Sigma^2_{11}}{\Delta_2}\end{pmatrix}+\Lambda^-\begin{pmatrix} \frac{-m_2+E-\Sigma^1_{22}}{\Delta_2} & 0\\ 0 & \frac{-m_1+E-\Sigma^1_{11}}{\Delta_1}\end{pmatrix}+$$

$$+\Lambda^+\gamma^5\begin{pmatrix} 0 & -\frac{\Sigma^3_{12}}{\Delta_1}\\ -\frac{\Sigma^3_{21}}{\Delta_2} & 0\end{pmatrix}+\Lambda^-\gamma^5\begin{pmatrix} 0 & -\frac{\Sigma^4_{12}}{\Delta_2}\\ -\frac{\Sigma^4_{21}}{\Delta_1} & 0\end{pmatrix}. \tag{45}$$

Here

$\Delta_1=\big(-m_1+E-\Sigma^2_{11}\big)\big(-m_2-E-\Sigma^2_{22}\big)-\Sigma^3_{12}\Sigma^4_{21}$,

$\Delta_2=\big(-m_1-E-\Sigma^1_{11}\big)\big(-m_2+E-\Sigma^1_{22}\big)-\Sigma^4_{12}\Sigma^3_{21}=\Delta_1\big(E\to -E\big)$.

The appearance of nilpotent operators in decomposition (45) is an indication for transitions between states of different parity.

Let us summarize our consideration of the dressing of Dirac fermions.

1) We found very convenient the using of the projection operators $\Lambda^\pm=\big(\sqrt{p^2}\pm\hat{p}\big)/2\sqrt{p^2}$ for solving of Dyson-Schwinger equation especially in case of few dressing states.

2) $\Lambda^\pm$ are very useful in another aspect: its coefficients have the definite parity. But as one can see from the loop calculations (28), (3.1) the components $\Lambda^\pm$ have different parity. There is such correspondence: the parity of the field Ψ is the parity of ,,alive“ component Λ^+, which has the pole $1/(E-m)$. Another component Λ^- which has the pole $1/(-E-m)$ demonstrates the opposite parity.

3) In contrast to boson case, even if the interactions conserve the parity, the loop transitions between different parity states are not zero: they are proportional to nilpotent operator $\mathcal{P}\mathcal{P} = 0$. $\Lambda^{\pm}\gamma^5$.

4) The joint dressing of two fermions without parity violation in vertex has different picture in dependence of parities of dressing states. One can illustrate it in the following scheme:

$$\mathbf{J^P = 1/2^{\pm}} \quad \Leftrightarrow \quad \mathbf{J^P = 1/2^{\pm}} \qquad \qquad \mathbf{J^P = 1/2^{\pm}} \quad \Leftrightarrow \quad \mathbf{J^P = 1/2^{\mp}}$$

$$\begin{array}{ccccccc} \Lambda^+ & \Longleftrightarrow & \Lambda^+ & \qquad & \Lambda^+ & \Longleftrightarrow & \Lambda^- \\ \Lambda^- & \Longleftrightarrow & \Lambda^- & \qquad & \Lambda^- & \Longleftrightarrow & \Lambda^+ \end{array}$$

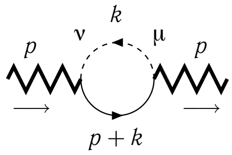
4 Spin-parity of the Rarita-Schwinger field

Compairing Tables 1 and 2, one can conclude that presence of the nilpotent operators $\mathcal{P}_7$—$\mathcal{P}_{10}$ in decomposition (15) is an indication on the transitions between components of different parity $1/2^{\pm}$. To make sure in this conclusion, we can calculate loop contributions in propagator. As an example we will take the interaction lagrangian $\pi N\Delta$

$$L_{int} = g_{\pi N\Delta}\ \overline{\Psi}^{\ \mu}(x)(g^{\mu\nu} + z\gamma^{\mu}\gamma^{\nu})\Psi(x)\cdot\partial_{\nu}\phi(x) + h.c.\ . \qquad (46)$$

Here z is so-called "off-shell" parameter.

The one-loop self-energy contribution is



$$J^{\mu\nu}(p) = -ig^2_{\pi N\Delta}\int\frac{d^4k}{(2\pi)^4}(g^{\mu\rho} + z\gamma^{\mu}\gamma^{\rho})k^{\rho}\frac{1}{\hat{p}+\hat{k}-m_N}k^{\lambda}(g^{\lambda\nu} + z\gamma^{\lambda}\gamma^{\nu})\frac{1}{k^2-m^2_{\pi}}. \qquad (47)$$

Let us calculate the discontinuity of loop contribution in $\hat{p}$ basis (11).

$$\Delta J_1 \ = \ -ig^2 I_0\frac{m_N}{12s}\lambda(s, m_N^2, m_\pi^2),$$

$$
\begin{aligned}
\Delta J_2 &= -ig^2 I_0 \frac{1}{24s^2}(s+m_N^2-m_\pi^2)\lambda, \\
\Delta J_3 &= -ig^2 I_0 \frac{m_N}{12s}(\lambda+6z\lambda-36z^2m_\pi^2 s), \\
\Delta J_4 &= -ig^2 I_0 \frac{1}{24s^2}[(s+m_N^2-m_\pi^2)\lambda+12zs\lambda+36z^2s(s^2-m_\pi^2 s-2m_N^2 s-m_\pi^2m_N^2+m_N^4)], \\
\Delta J_5 &= ig^2 I_0 \frac{m_N}{4s}[(s-m_N^2+m_\pi^2)^2+2z(s-m_N^2+m_\pi^2)^2+4z^2m_\pi^2 s], \\
\Delta J_6 &= ig^2 I_0 \frac{1}{8s^2}[(s+m_N^2-m_\pi^2)(s-m_N^2+m_\pi^2)^2+4zs(s-m_N^2+m_\pi^2)(s-m_N^2-m_\pi^2)+ \\
&\quad 4z^2s(s^2-m_\pi^2 s-2m_N^2 s-m_\pi^2m_N^2+m_N^4)], \\
\Delta J_7 &= ig^2 I_0 \sqrt{\frac{3}{s}}\cdot\frac{1}{24s}[(s-m_N^2+m_\pi^2)\lambda+4zs(2s^2-m_\pi^2 s-4m_N^2 s+2m_N^4-m_N^2m_\pi^2-m_\pi^4)+ \\
&\quad 12z^2s(s^2-m_\pi^2 s-2m_N^2 s-m_N^2m_\pi^2+m_N^4)], \\
\Delta J_8 &= -ig^2 I_0 \sqrt{\frac{3}{s}}\cdot\frac{zm_N}{6s}[(s^2+4m_\pi^2 s-2m_N^2 s+m^4-2m^2m_\pi^2+m_\pi^4)+6zsm_\pi^2], \\
\Delta J_9 &= \Delta J_7 \\
\Delta J_{10} &= -\Delta J_8.
\end{aligned}
\tag{48}
$$

Here I_0 is the base integral (3.1), $\lambda(a,b,c)=(a-b-c)-4bc$, arguments of λ are the same in all expressions, but are indicated only in first one.

We saw in above that in case of Dirac fermions the propagator decomposition in basis of projection operators demonstrates the definite parity. We can expect the similar property for Rarita-Schwinger field in Λ-basis. Let us verify it by calculating the threshold behavior of imaginary part. Using (48), one can convince yourself that

$$
\begin{aligned}
\Delta\bar{J}_1 &= \Delta J_1 + E\Delta J_2 \sim q^3 \\
\Delta\bar{J}_2 &= \Delta J_1 - E\Delta J_2 \sim q^5 \\
\Delta\bar{J}_3 &= \Delta J_3 + E\Delta J_4 \sim q^3 \\
\Delta\bar{J}_4 &= \Delta J_3 - E\Delta J_4 \sim q \\
\Delta\bar{J}_5 &= \Delta J_5 + E\Delta J_6 \sim q \\
\Delta\bar{J}_6 &= \Delta J_5 - E\Delta J_6 \sim q^3.
\end{aligned}
\tag{49}
$$

Such behavior indicates that the components $\bar{J}_1, \bar{J}_2$ exhibit the spin-parity $3/2^+$, while the pairs of coefficient $\bar{J}_3, \bar{J}_4$ and $\bar{J}_5, \bar{J}_6$ correspond to $1/2^+$ $1/2^-$ contributions respectively.

5 In conclusion:

- We obtained the general analytical expression (17) for the interacting Rarita-Schwinger field propagator which accounts for all spin components.
- The obtained dressed propagator (17) solves an algebraic part of the problem, the following step is renormalization. Note that the investigation of dressed propagator is the alternative for more conventional method based on equations of motion (see, *i.e.* Ref. [18] and references therein).
- We found that the nearest analogy for dressing of the $s = 1/2$ sector is the joint dressing of two Dirac fermions of different parity. Caculation of the self-energy contributions in case of Δ isobar confirms that in the Rarita-Schwinger field besides the leading $s = 3/2$ contribution there are also two $s = 1/2$ components of different parity.
- We suppose that such an approach is the more adequate for description of data on Δ production.

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